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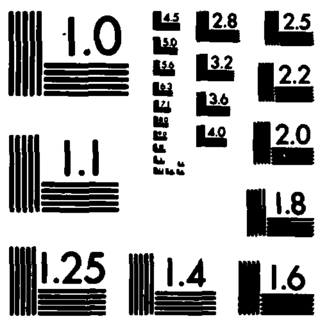

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IDENTIFIABILITY AND ESTIMATION IN

RANDOM TRANSLATIONS OF MARKED POINT PROCESSES[†]

by

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ABSTRACT

A random translation of a marked point process is considered. The random translation is assumed to be dependent upon the mark through a certain function, $h(\cdot)$. The main concern is to study the form of the function $h(\cdot)$ for different types of data. Complete identification is not generally possible but some interesting particular solutions are presented.

1. INTRODUCTION

1.1 Some Essential Definitions

Definition 1:

A point process, $\mathcal{I}$, is a stochastic process $\{I(t);t \geq 0\}$ defined over a fixed probability space $(\Omega,\mathcal{F},P)$ so that, for almost all $w \in \Omega$, the mapping $t \to I(w,t)$ is non-decreasing, right continuous, nonnegative, finite, integer valued, and has $I(w,0) = I(0) = 0$.

Such a process increases by jumps only, and the times of jumps will be called the "points" of $\mathcal{I}$. In particular, a point process, $\mathcal{I}$, with "points" $i_0, i_1, \ldots$ is called a renewal process if $i_0, i_1 - i_2, \ldots$ are independent and identically distributed (i.i.d.) random variables. A sequence $\{i_k ; k = 0,1, \ldots\}$ is usually called a realization of the point process ($i_k \in R_0^+$, the positive real numbers with zero included). A point process can be operated on and changed into another point process in several ways. Examples of operations on point processes are "superposition", "random deletion" and "random translation".

Definition 2:

A randomly translated point process $\mathcal{D}$, derived from an initial point process $\mathcal{I}$, is a point process with realizations $\{i_k + Y_k ; k = 0,1, \ldots\}$ where the sequence $\{Y_k ; k = 0,1, \ldots\}$ is a family of i.i.d. random variables and independent of the realization $\{i_k ; k = 0,1, \ldots\}$ of $\mathcal{I}$.

For more details about point processes, see Lewis [9].

A more general type of point process is the marked point process.

Definition 3:

A marked point process is a point process with an auxiliary variable, called the mark, associated with each point.

The mark can be a random variable, a random vector, or even a random process. The mark is used to identify random quantities associated with the point process it accompanies. An example of a mark is the velocity and acceleration of a vehicle passing an observer at a given instant. These processes were introduced by König and Matthes [7].

1.2 The Model

Let $\{I_k, L_k, Y_k \; ; \; k = 0,1, \ldots\}$ be three sequences of nonnegative continuous random variables. The first two random variables define the so called "input process", which in this case consists of the "arrival process" and "the marks" associated with it. The input process is assumed to be a marked point process and the arrival process is the corresponding point process. (The sequence $\{I_k \; ; \; k = 0,1, \ldots\}$ represents the inter-arrival times, that is to say the length of time between two consecutive jumps in the process $\{I(t); t \geq 0\}$, and $I_k = i_{k+1} - i_k$ for $k = 0,1, \ldots$ where i_0 is assumed to be zero.) The sequence $\{L_k \; ; \; k = 0,1, \ldots \; ; \; L_0 = 0\}$, represents the marks associated with the arrivals (L_k represents the mark of the k-th arrival). This sequence is a sequence of mutually i.i.d. random variables which are also independent of the instants of arrivals i_k's. The distribution function of the marks is denoted by $P(x)$. The random variables, Y_k's, define the random translations (Y_k represents the random translation of the k-th arrival which has occurred at the instant i_k with the mark L_k). The sequence $\{Y_k \; ; \; k = 0,1, \ldots \; ; \; Y_0 = 0\}$ is assumed

to be a sequence of i.i.d. random variables which are independent of the instants of arrival. In this paper, we assume the existence of a certain function $h(\cdot)$ such that the random variables Y_k's depend on the marks in the following way:

$$P[Y_k \leq x \mid L_k = \ell] = \begin{cases} 1 - e^{-h(\ell)x} & , \ x \geq 0 \\ 0 & , \ x < 0 \end{cases} \quad \text{for } k = 1,2, \ldots . \tag{1}$$

The $h(\cdot)$ is a function relating the mark of the arrival to the parameter of the exponential distribution which define the conditional distribution function of the random variable Y_k. Let d_k be the point of the translated point process $\mathcal{D}$, corresponding to the point i_k of the point process $\mathcal{I}$. By Definition 2, we have:

$$d_k = i_k + Y_k \quad \text{for } k = 0,1, \ldots . \tag{2}$$

The sequence $\{D_k = d_{k+1} - d_k \,;\, k = 0,1, \ldots\}$ represents the time between two consecutive points of the translated point process $\mathcal{D}$.

The mathematical model defined above is a random translation of a marked point process.

1.3 Principal Results

Several questions of a probabilistic and statistical kind can be asked in relation to this model. In this paper, our main concern is to study the form of the function $h(\cdot)$ for different types of data.

Complete identification is not generally possible, but it will be shown, for example, that maximum likelihood estimators (m.l.e) can be found for some interesting particular situations. In Section 2, a formal solution

is presented where the study of the form of $h(\cdot)$ is related to the problem of identifiability and estimation of the service time distribution function of the $G/G/\infty$ system. For uniformly distributed marks, a solution is given. In Section 3, the input process is restricted to a marked non-homogeneous Poisson process and it is proved that the total number of transitions (arrivals and departures from the $\bar{M}/G/\infty$ system, regardless of their marks) during a fixed time interval $(0,t]$ is a Hermite random variable. This result is of independent interest and holds for any infinite server system with non-homogeneous Poisson arrival process and general service time distribution. The form of the function $h(\cdot)$ is assumed to be known, apart from a certain number of unknown parameters, and m.l.e. can be found for these parameters and the basic parameters of the arrival process when working with not more than two unknown parameters. Moreover, the data required by these estimators is just the total number of indistinguishable arrivals and departures from the system during a fixed time interval. A discussion will be found in Sections 4 and 5 concerning other types of data, namely, observations just of the instants of departures during time interval $(0,t]$ and direct observations of the service times.

1.4 The Corresponding $G/G/\infty$ System

If we don't know the marks of the arrivals, our model can be regarded as a $G/G/\infty$ system where the arrival process is the same but the distribution function of the service time, $G(x)$, is given by:

$$G(x) = P[Y_j \leq x]$$

$$= \int_0^{+\infty} P[Y_j \leq x \mid \ell_j = \ell] dP(\ell)$$

$$= \begin{cases} 1 - \int_0^{+\infty} e^{-h(\ell)x} dP(\ell) & , \quad x \geq 0 \\ 0 & , \quad x < 0 . \end{cases} \tag{3}$$

This service time distribution function depends heavily on the particular type of the distribution function of the marks, $P(\cdot)$, and on the form of the function $h(\cdot)$. For instance, if $h(\cdot)$ is a step function, for any distribution function of the marks, $G(\cdot)$ will be the distribution function of a mixture of exponentials. Or, if the function $h(\cdot)$ and the random variable L are such that the random variable $h(L)$ is normally distributed with mean zero and variance 1/2, then $G(x) = \begin{cases} 1 - e^{-x^2} & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$ and the service time, Y , will be a Weibull random variable.

We call this $G/G/\infty$ system the "corresponding $G/G/\infty$ system" and denote by $\tilde{M}/G/\infty$ the "corresponding $G/G/\infty$ system" with a non-homogeneous Poisson arrival process.

2. FORMAL SOLUTION FOR THE IDENTIFICATION-ESTIMATION PROBLEM

We have seen that regardless of the marks our model is equivalent to a $G/G/\infty$ system with service time distribution function given by (3), i.e., the "corresponding $G/G/\infty$ system". On the other hand,

$$\bar{G}(x) = 1 - G(x) = P[Y_j > x]$$

$$= \begin{cases} E[e^{-h(L)x}] & , \quad x \geq 0 \\ 1 & , \quad x < 0 , \end{cases} \tag{4}$$

assuming that $h(\cdot)$ is such that $h(L)$ is a random variable.

<u>Theorem 1</u>:

If the random variable $h(L)$ is positive and absolutely continuous, i.e., $dF_{h(L)}(y) = f_{h(L)}(y)dy$, if $f_{h(L)}(y) \in C^1$ for $y > 0$ and if $\bar{G}(x) \in C^{\infty}$ (a) in a neighborhood (however small) of $+\infty$. Then,

$$f_{h(L)}(y) = \lim_{k\to\infty} L_{k,y}[\bar{G}(x)] \quad , \quad y > 0 , \tag{5}$$

where $L_{k,y}[\bar{G}(x)] = \frac{(-1)^k}{k!}\left(\frac{k}{y}\right)^{k+1}\bar{G}^{(k)}\left(\frac{k}{y}\right)$ (b) is an operator defined for any positive real number y and any large integer k.

<u>Proof</u>:

By (4) and by the assumption that $h(L)$ is an absolutely continuous positive random variable, we get:

$$\bar{G}(x) = \begin{cases} \int_0^{+\infty} e^{-yx} f_{h(L)}(y)dy & , \quad x \geq 0 \\ 1 & , \quad x < 0 \end{cases} \tag{6}$$

i.e., $\bar{G}(\cdot)$ is the Laplace transform of $f_{h(L)}(\cdot)$. The result thus follows by a direct use of a result in Widder [16, real inversion of the Laplace transform; pp. 140 and 141].

Furthermore, if we assume that $P(\cdot)$ is also known, then at least theoretically we can evaluate $h(\cdot)$. For example, if the marks, L , are assumed to be uniform random variables in [0,1] and if $h(\cdot)$ is assumed to be a non-decreasing function defined on [0,1] , then

$$\begin{aligned} h(y) &= F^{-1}_{h(L)}(y) \\ &= \{x : \text{the smallest value of } x \text{ so that } F_{h(L)}(x) \le y\} , \\ &\quad 0 \le y \le 1 . \end{aligned} \tag{7}$$

<u>Remark 1</u>:

If $h(L)$ is a discrete random variable, then its distribution function $F_{h(L)}(\cdot)$ is a step function and $\int_0^{+\infty} e^{-yx} dF_{h(L)}(y) = \bar{G}(x)$ reduces to a Dirichelet series; see Widder [16, pp. 93 and 94].

<u>Remark 2</u>:

By Theorem 3.3 in Widder [16, p. 203] with $\alpha = 1$, we get the following result:

If $\bar{G}(x) \sim \frac{A}{x}$ as $x \to 0^+$,

then $\lim_{x \to +\infty} \int_0^x t^{\beta-1} f_{h(L)}(t)dt = E[\{h(L)\}^{\beta-1}] \sim \frac{Ax^\beta}{\beta}$;

where $\beta = 2,3, \ldots$.

Remark 3:

There are several ways of estimating the service time distribution function, $G(\cdot)$, according to the data available. For example, if the data consist of random observations of the service times, $y_1, \ldots, y_n$, then the empirical distribution function, $G_n(y_1, \ldots, y_n)$, is a consistent estimator of $G(x)$. Brillinger [2] obtained an estimator of the service time density function, $g(x) = -\frac{d}{dx}\bar{G}(x)$, based on a stretch of arrival and departure times (arrival-departure records) for the stationary $G/G/\infty$ system (having independent service times with a common finite mean value) and Ross [14] shows that, for the $GI/G/k$ system with $k \geq \infty$, $G(x)$ is identifiable from a single sample path of the process $\{X(t), t \geq 0\}$.

3. THE HERMITE DISTRIBUTION APPROACH

Let the point process $\{A(t);t \geq 0\}$ have the following properties:

(i) $E\{A(t)\} = m(t) < \infty$, $t \geq 0$,

(ii) $\frac{dm(t)}{dt} =$ (t) > 0 and bounded on finite intervals for $t > 0$.

Definition 4:

A point process $\{A(t);t \geq 0\}$ has the order statistics property if and only if, given that n jump times (points) occur in $(0,t]$, the conditional distribution of $i_1, \ldots, i_n$ is the same as the distribution of the order statistics of a random sample of size n from the distribution with density function,

$$f_t(x) = \frac{\lambda(x)}{m(t)} \quad , \quad 0 \leq x \leq t \ . \tag{8}$$

The Poisson process is the most familiar example of a process with this property, in which $f_t(x) = \frac{1}{t}$ for $x \in [0,t]$ (i.e., uniform distribution in $[0,1]$). However, this does not characterize the Poisson process since, for example, the number of births in a linear birth process with parameter $\lambda > 0$ also has the order statistics property with $f_t(x) = \frac{\lambda e^{\lambda x}}{e^{\lambda t} - 1}$ for $x \in [0,t]$, see Neuts and Resnick [12]. It can be shown as well, that the non-homogeneous Poisson process also has the order statistics property; see, for example, Snyder [15, p. 65]. Let $m(t)$, the mean value function of the non-homogeneous Poisson process, be an absolutely continuous function of t , and $\frac{dm(t)}{dt} = \lambda(t)$ its intensity function. Then,

$$f_t(x) = \frac{\lambda(x)}{m(t)} \quad , \quad 0 \leq x \leq t \ . \tag{9}$$

(For more details on point processes with the order statistics property, see also Feigin [5].)

In our general model, let $I(t)$ represent the number of arrivals, regardless of the marks, during time interval $(0,t]$ and $D(t)$ the number of departures during time interval $(0,t]$, regardless of the marks, among the ones arriving during $(0,t]$.

Also let $X(t)$ represent the number of arrivals, regardless of the marks, still present in the system at time t, among the ones arriving during time interval $(0,t]$ and let $N(t)$ represent the number of transitions, arrivals and departures regardless of their marks, during a fixed time interval $(0,t]$.

<u>Theorem 2</u>:

For the $\bar{M}/G/\infty$ system and for each fixed time interval $(0,t]$, the random variable $N(t)$ is a Hermite variable with probability function given by:

$$P[N(t) = n] = e^{-m(t)} \sum_{j=0}^{[\frac{1}{2}n]} \frac{[a_1(t)]^{n-2j}}{(n-2j)!} \frac{[a_2(t)]^j}{j!} \tag{10}$$

for $n = 0,1, \ldots$; where $[b]$ means the integer part of b ; $a_1(t) = m(t)(1-p)$; $a_2(t) = m(t)p$ and $p = p(t,h(\cdot),P(\cdot),m(\cdot)) = \frac{1}{m(t)} \int_0^t \left[\int_0^{+\infty} e^{-h(\ell)(t-u)} dP(\ell)\right] dm(u)$.

<u>Proof</u>:

For each fixed time interval $(0,t]$, we have for any $G/G/\infty$ system

$$N(t) = I(t) + D(t) , \tag{11}$$

and,

$$N(t) = X(t) + 2D(t) \ . \tag{12}$$

We shall first prove that the random variables $X(t)$ and $D(t)$ are two independent Poisson random variables and then use this result and the probability generating function of $N(t)$, $E[z^{N(t)}]$, to complete the proof of the theorem.

For each fixed time interval $(0,t]$, given that $I(t)$ arrivals had occurred, the order statistics property of the input process and the assumption that the random variable Y associated with each arrival is independent of everything else in the system (apart from the mark carried out by this arrival) allow us to write $X(t)$ in the following way:

$$X(t) = W_1(t) + \ldots + W_{I(t)}(t) \ .$$

Where the $W_j(t)$'s are a sequence of independent Bernoulli random variables defined as follows:

$$W_j(t) = \begin{cases} 1 \ , & \text{if the j-th arrival which has occurred at time} \\ & i_j \in [0,t] \text{ , is still in the system at time } t \\ & \\ 0 \ , & \text{if the j-th arrival which has occurred at time} \\ & i_j \in [0,t] \text{ , is no longer in the system at time} \\ & t \end{cases} \tag{13}$$

for $j = 1, \ldots, I(t)$. The parameter of the Bernoulli distribution is given by:

$$P[W_j(t) = 1] = \int_0^t P[Y_j > t - u] f_t(u)du$$

$$= \frac{1}{m(t)} \int_0^t [1 - G(t - u)]\lambda(u)du$$

$$= \frac{1}{m(t)} \int_0^t \left[\int_0^{+\infty} e^{-h(\ell)(t-u)} dP(\ell)\right] \lambda(u)du$$

$$= p(t,h(\cdot),P(\cdot),m(\cdot)) \quad \text{for} \quad j = 1, \ldots, I(t) \tag{14}$$

which is independent of j. For the sake of simplicity, we shall use "p" instead of $p(t,h(\cdot),P(\cdot),m(\cdot))$. Therefore, the sequence $\{W_j(t) ; j = 1, \ldots, I(t)\}$ is a sequence of i.i.d. Bernoulli random variables of parameter p. Then, for each fixed $t > 0$, the probability function of the random variable $X(t)$ is given by:

$$P[X(t) = k] = \sum_{n=k}^{\infty} P[X(t) = k \mid I(t) = n] P[I(t) = n]$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1 - p)^{n-k} P[I(t) = n]$$

$$= \frac{p^k}{k!} \sum_{n=k}^{\infty} n! \frac{(1 - p)^{n-k}}{(n - k)!} P[I(t) = n] \quad \text{for} \quad k = 0,1, \ldots . \tag{15}$$

In the case of a non-homogeneous Poisson process input, the expression (15) reduces to:

$$P[X(t) = k] = e^{-pm(t)} \frac{[pm(t)]^k}{k!} \quad \text{for} \quad k = 0,1, \ldots . \tag{16}$$

What we have done is to associate a Bernoulli random variable with each arrival that occurred during the time interval $(0,t]$. If we now inter-

change the role of "0" and "1" (success, failure) in the Bernoulli variables, $W_j(t)$, and follow exactly the same steps as above, we obtain the probability function of the random variable $D(t)$, which is the same as the probability function of $X(t)$ with p replaced by $(1 - p)$ and $(1 - p)$ replaced by p :

$$P[D(t) = k] = \sum_{n=k}^{\infty} P[D(t) = k \mid I(t) = n]P[I(t) = n]$$

$$= \sum_{n=k}^{\infty} \binom{n}{k}(1 - p)^k p^{n-k} P[I(t) = n]$$

$$= \frac{(1 - p)^k}{k!} \sum_{n=k}^{\infty} n! \frac{p^{n-k}}{(n - k)!} P[I(t) = n] \quad \text{for } k = 0,1, \ldots . \tag{17}$$

In the case of a non-homogeneous Poisson input, expression (17) reduces to:

$$P[D(t) = k] = e^{-(1-p)m(t)} \frac{[(1 - p)m(t)]^k}{k!} \quad \text{for } k = 0,1, \ldots . \tag{18}$$

In the case of non-homogeneous Poisson input, the independence of the random variables $X(t)$ and $D(t)$ is well known and easy to prove. Therefore,

$$E[z^{N(t)}] = E[z^{X(t)}]E[z^{2D(t)}]$$

$$= e^{m(t)p(z-1)} \times e^{m(t)(1-p)(z^2-1)}$$

$$= e^{[m(t)p(z-1)+m(t)(1-p)(z^2-1)]} . \tag{19}$$

This probability generating function is known to be the probability generating function of a Hermite random variable. In this case with parameters, $a_1(t) = m(t)(1 - p)$ and $a_2(t) = m(t)p$ where by (14) $p = p(t,h(\cdot),P(\cdot)$,

$m(\cdot)) = \frac{1}{m(t)} \int_0^t \left[\int_0^{+\infty} e^{-h(\ell)(t-u)} dP(\ell) \right] dm(u)$. Therefore, the result follows, i.e., the probability function of $N(t)$ is:

$$P[N(t) = n] = e^{-m(t)} \sum_{j=0}^{[\frac{1}{2}n]} \frac{[a_1(t)]^{n-2j}}{(n - 2j)!} \frac{[a_2(t)]^j}{j!} \ ; \tag{10}$$

$n = 0,1, \ldots$.

We notice that this theorem provides us with two results of independent interest concerning infinite server systems:

- the theorem is valid for any infinite server system of non-homogeneous Poisson arrival process and general service time distribution function.

- in the case of an infinite server system of general service time distribution and a "point process with the order statistics property" as the arrival process, the random variable $N(t)$ does not seem to be Hermite. Nevertheless, the distributions of $X(t)$ and $D(t)$ still show a very similar structure between themselves. The only difference is the interchange role of p and $(1 - p)$; see formulae (15) and (17).

The Hermite distribution is a well studied random variable with two parameters $a_1(t)$ and $a_2(t)$; m.l.e. for these parameters can be found in Kemp and Kemp [6, pp. 386 to 388]. We can now establish the following result:

<u>Corollary</u>:

If the form of the function $h(\cdot)$ is assumed to be known apart from a certain number of unknown parameters, then the m.l.e. $\hat{a}_1(t)$ and $\hat{a}_2(t)$

of the Hermite distribution, together with the two equations obtained in Theorem 2, i.e., for each fixed $t \geq 0$,

$$\begin{cases} a_2(t) = \int_0^t \left[\int_0^{+\infty} e^{-h(\ell)t-v} dP(\ell) \right] dm(v) \\ a_1(t) = m(t) - a_2(t) \end{cases} \tag{20}$$

allow us to obtain m.l.e. for the unknown parameters of the function $h(\cdot)$ and for the unknown parameters of the mean value function, $m(t)$, of the arrival process, when working with not more than two unknown parameters.

We note that the data required is just the total number of indistinguishable arrivals and departures from the system during a fixed time interval.

To illustrate the method suggested by this corollary, we present the following example of an application:

Example 1:

Let $m(t) = \lambda t$ and $h(\cdot)$ be a Heaviside function, i.e.,

$$h(\ell) = \begin{cases} \mu_1 , & 0 \leq \ell \leq \ell_0 \\ \mu_2 , & \ell > \ell_0 \end{cases} \tag{21}$$

where μ_1 , μ_2 are known positive real numbers. Let L be a random variable with known and invertible distribution function $P(x)$. The m.l.e. $\hat{\lambda}$, $\hat{\ell}_0$ of the unknown parameters λ and ℓ_0 are given in terms of the m.l.e. $\hat{a}_1(t)$ and $\hat{a}_2(t)$ in the following way:

$$\begin{cases} \hat{\lambda} = \dfrac{\hat{a}_1(t) + \hat{a}_2(t)}{t} \\ \hat{l}_0 = P^{-1}\left(\dfrac{\mu_1\mu_2 t}{\mu_2(1 - e^{-\mu_1 t}) - \mu_1(1 - e^{-\mu_2 t})} \times \right. \\ \left. \left[\dfrac{\hat{a}_2(t)}{\hat{a}_1(t) + \hat{a}_2(t)} - \dfrac{1 - e^{-\mu_2 t}}{\mu_2 t}\right]\right). \end{cases} \tag{22}$$

<u>Solution</u>:

By Theorem 2, for each fixed $t > 0$,

$$\begin{aligned} a_2(t) &= m(t)p \\ &= \int_0^t \left[\int_0^{+\infty} e^{-h(\ell)(t-u)}\, dP(\ell)\right] dm(u) \\ &= \lambda \int_0^t \left[p_0 e^{-\mu_1(t-u)} + (1 - p_0)e^{-\mu_2(t-u)}\right] du \end{aligned} \tag{23}$$

where $p_0 = P(\ell_0) = P[L \le \ell_0]$. And $a_1(t) = \lambda t - a_2(t)$. Therefore, if $\hat{a}_1(t)$ and $\hat{a}_2(t)$ represent the m.l.e. of $a_1(t)$ and $a_2(t)$ respectively, then

$$\lambda = \frac{\hat{a}_1(t) - \hat{a}_2(t)}{t}$$

$$p_0 = \frac{\mu_1\mu_2 t}{\mu_2(1 - e^{-\mu_1 t}) - \mu_1(1 - e^{-\mu_2 t})}\left[\frac{\hat{a}_2(t)}{\hat{a}_1(t) + \hat{a}_2(t)} - \frac{1 - e^{-\mu_2 t}}{\mu_2 t}\right].$$

(Because p_0 represents a probability, $p_0 = P[L \leq \ell_0]$, we have to have $0 \leq p_0 \leq 1$; therefore, μ_1 , μ_2 and the m.l.e. $\hat{a}_1(t)$ and $\hat{a}_2(t)$ have to be such that the random variable $\hat{p}_0$ will take values between zero and one.)

On the other hand, if the distribution function of L , $P(x)$, is invertible, then $p_0 = P[L \leq \ell_0] = p(\ell_0)$ and $p_0 = P^{-1}(p_0)$, i.e.,

$$\lambda = \frac{\hat{a}_1(t) + \hat{a}_2(t)}{t}$$

$$\hat{\imath}_0 = P^{-1}(\hat{p}_0)$$

where $\hat{\ell}_0$ represents the m.l.e. of the parameter ℓ_0 .

4. DIRECT OBSERVATIONS ON THE DEPARTURE PROCESS

Let the data consist of observations on the departure process, namely the number of departures, $D(t)$, from the system and the instants of departure $d_1, \ldots, d_{D(t)}$, during a fixed time interval $[0,t]$. Let the input process be a marked non-homogeneous Poisson process, of known mean value function $m(t)$. The following remarks in the context of infinite server queues will lead us to the evaluation of some particular solutions of the parametric and non-parametric type. First, by a well known result in the general $\tilde{M}/G/\infty$ system, due to Mirasol [11], the corresponding departure process, regardless of the marks, is a non-homogeneous Poisson process of mean value function $m_D(t)$ (in the stationary case $m_D(t) = m(t)$). Secondly, the result (18) permits us to express $m_D(t)$ in terms of $m(\cdot)$, $h(\cdot)$ and $P(\cdot)$.

$$
\begin{aligned}
m_D(t) &= m(t)(1 - p) \\
&= m(t) - \int_0^t \bar{G}(t-u)\,dm(u) \\
&= m(t) - \int_0^t \left[\int_0^{+\infty} e^{-h(\ell)(t-u)}\,dP(\ell)\right] dm(u) \\
&= \int_0^t E[1 - e^{-h(L)(t-u)}]\,dm(u) \ .
\end{aligned} \tag{24}
$$

The following examples illustrate the type of parametric solution that we can get from (24).

Example 1:

Let the input process be a marked Poisson process, the random variables L_k's be i.i.d. and uniform in $[0,1]$, and let the function $h(\cdot)$ be $h(\ell) = -b \log \ell$. The mean value function $m_D(t)$ is given by:

$$m_D(t) = \lambda t - \lambda \int_0^t \left[\int_0^1 e^{(\log \ell) b(t-u)} dP(\ell) \right] du$$

$$= \lambda t - \lambda \int_0^t \frac{1}{(1 + bt) - bu} du$$

$$= \lambda t - \lambda \frac{\log (bt + 1)}{b} . \tag{25}$$

If $\hat{m}_D(t)$ is a m.l.e. of $m_D(t)$, the random variable $\hat{b}$ defined by:

$$\frac{\log (\hat{b}t + 1)}{\hat{b}} = \frac{\lambda t - \hat{m}_D(t)}{\lambda} \tag{26}$$

is a m.l.e. of the parameter b .

The result (18) tell us that, for each fixed time interval $[0,t]$, the random variable number of departures from the system, $D(t)$, is a Poisson random variable of mean value $m_D(t)$ and then that the number of departures during $[0,t]$ is a m.l.e. of $m_D(t)$.

Example 2:

If in the above example the mark L is assumed to be a random vector, i.e., $L = (L_1, \ldots, L_c)$ with the components L_i for $i = 1, \ldots, c$ random variables i.i.d. and uniformly distributed in $[0,1]$. And if $h(\cdot)$

is assumed to be of a similar form as in the Example 1 above, i.e., $h(L) = -b \log\left(\prod_{i=1}^{c} L_i\right)$, then we get in a similar manner a m.l.e. for the unknown parameter b.

$$m_D(t) = \lambda t - \lambda \int_0^t \left[\int_0^1 e^{(\log \ell) b(t-u)} dP(\ell) \times \dots \times \int_0^1 e^{(\log \ell) b(t-u)} dP(\ell)\right] du$$

$$= \lambda t - \lambda \int_0^t \frac{1}{(b(t-u)+1)^c} du \quad ; \quad c = 2,3, \dots \tag{27}$$

and

$$\hat{m}_D(t) = \lambda t - \frac{\lambda[(\hat{b}+t)^{1-c} - 1]}{\hat{b}(1-c)} \quad ; \quad c = 2,3, \dots \tag{28}$$

(where as before $\hat{m}_D(t)$ is the random variable, number of departure sampled during the fixed time interval $[0,t]$).

Some Comments on the Non-parametric Case

As before let the input be a marked non-homogeneous Poisson process of known intensity function $\lambda(t)$ and let $\mu(t)$ be the intensity function of the departure process, $\mu(t) = \frac{d}{dt} m_D(t)$;

$$\mu(t) = \lambda(t) - \frac{d}{dt} \int_0^t \left[\int_0^{+\infty} e^{-h(\ell)(t-u)} dP(\ell)\right] \lambda(u) du \ . \tag{29}$$

If $P(\cdot)$ is assumed to be known and the integrals involved exist and are finite, then $\mu(t) = H(t, h(\cdot))$ where the function H is known and given by (29). Therefore, if the function $\mu(t)$ or an estimator of it is known,

at least theoretically, we should be able to detect the structure of $h(\cdot)$ from (29). Similarly, if $h(\cdot)$ and $\mu(t)$ are assumed to be known, then we can evaluate exact or approximately the function $P(\cdot)$ from (29).

There are several approaches to the problem of estimating the intensity function $\mu(t)$, of a non-homogeneous Poisson process (departure process of the $\bar{M}/G/\infty$ system) from the number of occurrences in a fixed time interval, $D(t)$, and their corresponding occurrence times, $t_1, \ldots, t_{D(t)}$. (For a completely non-parametric approach see Cleverson and Zidek [3], where $\mu(t)$ is only required to be positive and integrable on $[-T,T]$ for $0 < T \leq +\infty$, and Leonard [8, pp. 121 to 123]. Both authors make use of the order statistics property, defined on page 9, of the non-homogeneous Poisson process to relate the problem of estimating Poisson intensity functions to the problem of estimating probability density functions. See also Cox [4] and Snyder [15].)

5. DIRECT OBSERVATIONS ON THE LENGTH OF THE SERVICE TIMES

5.1 Direct Observations of the Length of the Service Times Regardless of Their Corresponding Marks

Let the data available consist of direct observations of the length of the service times during a fixed time interval $(0,t]$. The expression (3)

$$\bar{G}(x) = 1 - G(x)$$

$$= \begin{cases} \int_0^{+\infty} e^{-h(\ell)x} dP(\ell) & , x \geq 0 \\ 1 & , x < 0 , \end{cases}$$

is valid for an input process which is any general marked point process and allows us to solve the following two types of estimation problems.

Parametric Estimation of $h(\cdot)$ and $P(\cdot)$

If $h(\cdot)$ and $P(\cdot)$ belong to a certain known family of functions with some unknown parameters, the expression (3) provides a method to obtain m.l.e. for these unknown parameters.

Theorem 3:

Let the arrival process of the "corresponding $G/G/\infty$ system" be any general point process. Let $h(\cdot)$ and $P(\cdot)$ be known functions except for a certain number of unknown parameters. Let the data available consist of direct observations of the length of the service times. Then the expression (3), $\bar{G}(x) = \begin{cases} \int_0^{+\infty} e^{-h(\ell)x} dP(\ell) & , x \geq 0 \\ 1 & , x < 0 \end{cases}$, allows us to obtain m.l.e. for the unknown parameters of $h(\cdot)$ and $P(\cdot)$.

Proof:

By hypothesis, $h(\cdot)$ and $P(\cdot)$ are known functions of the parameters $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ respectively. And the density function of the service times, $g(\cdot)$, is $g(x) = -\frac{d\bar{G}(x)}{dx}$ for $x \geq 0$. Therefore, the likelihood function of any random sample of observed service times $(y_1, \ldots, y_k)$ is given by:

$$L(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_n; y_1, \ldots, y_k) = \prod_{i=1}^{k} g(y_i)$$

$$= \prod_{i=1}^{k} \int_0^{+\infty} h(\ell) e^{-h(\ell) y_i} dP(\ell) \ . \tag{30}$$

The maximum likelihood method can be applied, in a very straightforward way, to get m.l.e. for $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$.

Example 1:

Let $h(\cdot)$ be the Heaviside function defined in (21) with ℓ_0, μ_1 and μ_2 unknown parameters. By replacing $h(\cdot)$ by this Heaviside function in (3), we get:

$$\bar{G}(x) = \begin{cases} p_0 e^{-\mu_1 x} + (1 - p_0) e^{-\mu_2 x} & ; \quad x \geq 0 \\ 1 & ; \quad x < 0 \end{cases} \tag{31}$$

where $p_0 = P[L \leq \ell_0] = P(\ell_0)$ and the density function of the service time, $g(x)$, is

$$g(x) = \mu_1 p_0 e^{-\mu_1 x} + \mu_2 (1 - p_0) e^{-\mu_2 x} \quad ; \quad x \geq 0 \ . \tag{32}$$

Let $(y_1, \ldots, y_n)$ be a realization of a random sample of the length of the service time. A direct application of the maximum likelihood estimation method $\left(\frac{\partial L(x)}{\partial \mu_1} = 0 ; \frac{\partial L(x)}{\partial \mu_2} = 0 ; \frac{\partial L(x)}{\partial p_0} = 0 \text{ where } L(x) = \sum_{i=1}^{n} \log \left[\mu_1 p_0 e^{-\mu_1 y_i} + \mu_2(1 - p_0)e^{-\mu_2 y_i}\right]\right)$ will provide us with the m.l.e. $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{p}_0$. If the distribution function of L , $P(\cdot)$, is known and invertible, then $\hat{\ell}_0 = P^{-1}(\hat{p}_0)$.

More complicated forms of $h(\cdot)$ could be envisaged and the possibility of parametrically estimating $h(\cdot)$ and also $P(\cdot)$ exist; whether this is possible depends on our identifiability problem (chosen shape of $h(\cdot)$) and on the nature of the known distribution function $P(\cdot)$ (convergence of the improper integrals). But the m.l.e. of the parameters involved may be difficult or impossible to evaluate analytically, and then they have to be obtained numerically.

In comparison with the method presented in Section 3, the present method is much less restrictive, because the input can be any general marked point process and more than two parameters can be estimated from the data. In fact, we can obtain m.l.e. for as many parameters as we wish provided that the improper integrals involved will converge and that "$1 - \bar{G}(\cdot)$" actually defines a distribution function. However, a more complex type of data is required and nothing can be learned about the arrival process.

Non-Parametric Estimation of $P(\cdot)$

Let $h(\cdot)$ be a completely known function. Can we get a density estimator for the distribution function of the marks from the direct observations of the length of the service time? The expression (3) converts this

problem to a typical empirical Bayesian estimation problem; see, for instance, Robbins [13].

As a matter of fact, the expression (3) can be rewritten as follows:

$$G(x) = P[Y \le x] = \int_0^{+\infty} P[Y \le x \mid L = \ell]\,dP(\ell)$$

$$= \int_0^{+\infty} G_{h(\ell)}(x)\,dP(\ell) \quad , \quad x \ge 0 . \tag{33}$$

In the terminology of the empirical Bayesian theory, $G(x)$ is the "mixed" distribution, $P(\cdot)$ is a "mixing" distribution, and $G_{h(\ell)}(\cdot)$ is the known kernel. This theory provides us with a method to obtain a consistent estimator of the distribution function of the marks, $P_n(\cdot)$.

By using the empirical distribution function, $G_n(\cdot)$:

$$G_n(x) = \frac{\text{no. of terms } y_1, y_2, \ldots, y_n \text{ which are } \le x}{n} \tag{34}$$

(which converges uniformly to $G(\cdot)$ with probability 1 as $n \to +\infty$) and the functional equation (33) it is possible, at least formally, to obtain a distribution function $P_n(\cdot)$ which will converge as $n \to +\infty$ to the unknown distribution function $P(\cdot)$. The degree of difficulty in getting this estimator $P_n(\cdot)$ is very much related to the nature of the known kernel, which here we denoted by $G_{h(\ell)}(\cdot)$. In our case, $G_{h(\ell)}(x) = 1 - e^{-h(\ell)x}$. Our problem can be generalized by considering that the conditional distribution function of the service time of each arrival is any other specified distribution (with the basic parameters to be known functions of the marks)

rather than the exponential distribution as assumed before. For more details on empirical Bayesian methods, see Maritz [10].

Non-Parametric Estimation of $h(\cdot)$

If $P(\cdot)$ is completely known, an estimator of $h(\cdot)$ can be obtained by using the results in Section 2.

5.2 Direct Observations on the Length of the Service Times and Their Corresponding Marks

Let us now assume that a direct observation of the marks as well as of the length of the service times can be obtained. In medical and reliability data very often this type of data is available. The data now consist of n pairs of positive real numbers $((y_1,\ell_1),(y_2,\ell_2), \ldots, (y_n,\ell_n))$ where the first number in the pair represents the length of time of the service and the second represents the corresponding mark. The likelihood function is then given by:

$$L(y_1,\ell_2;y_2,\ell_2; \ldots; y_n \ell_n) = \prod_{i=1}^{n} h(\ell_i)e^{-h(\ell_i)y_i} \tag{42}$$

and

$$\log L(y_1,\ell_1;y_2,\ell_2; \ldots; y_n,\ell_n) = \sum_{i=1}^{n} \log h(\ell_i) - \sum_{i=1}^{n} h(\ell_i)y_i \; . \tag{43}$$

Let $\ell_{i_1}, \ldots, \ell_{i_k}$ for $i_k \leq n$ be the k distinct values of ℓ among the n observed. Let $y_{1,i_j}, \ldots, y_{r_{(j)},i_j}$ for $r_{(j)} \leq n$ and $j = 1, \ldots, k$ and $\sum_{j=1}^{k} r_{(j)} = n$ be the values of the observed service

times, y's , of mark ℓ_{i_j} . Let $\bar{y}_j = \frac{\sum_{s=1}^{r_{(j)}} y_{s,i_j}}{r_{(j)}}$ be its sample average service time. In our model given the value of the mark, for instance ℓ_{i_j} , the corresponding length of the service time is an exponential random variable of parameter $\mu_j = h(\ell_{i_j})$. The m.l.e. of the parameter μ_j is the inverse sample average $(\bar{y}_j)^{-1}$. Therefore, if in our sample we plot the values ℓ_{i_j} for $j = 1, \ldots, k$ against the corresponding inverse sample averages $(\bar{y}_j)^{-1}$, we get some empirical information about the form of the function $h(\cdot)$. Thus to estimate $h(\cdot)$ with this type of data we suggest to use data in two stages: first, to learn about the form of the function in the way presented above and to choose a suitable parametric representation for $h(\cdot)$ and secondly to obtain through (43) m.l.e. for the parameters of $h(\cdot)$.

We note that if in our model the marks are ℓ_i , $i = 1,2$ and if they are interpreted as denoting respectively "failure" and "censoring" or "withdrawn" in the context of reliability theory then the Bayesian statistical results presented in Barlow and Proschan [1] can be used (in their paper the underlying probabilistic model is of the $G/G/\infty$ type; the observed individuals enter into the system, stay there for a certain time (lifetime) and then leave without interacting among themselves).

FINAL REMARKS

The conditional distribution of the service time, $P[Y_k \leq x \mid L_k = \ell]$, do not need to be exponential, except for Theorem 1.

No attempt has been made to study the performance of the various types of estimators presented and suggested in this paper. The reason is that the problem is too complicated and we are just beginning to understand it. However, its close relationship with areas traditionally considered to be unrelated has been noted in this paper. To our mind, these relationships deserve further attention.

Footnotes

(a) C^{∞} - means the class of the continuous functions with all derivatives.

(b) $\bar{G}^{(k)}$ - means the k-th derivative of $\bar{G}$.

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